# Introduction to Differential Equations

# ordinary differential equations

**Definition**:

A differential equation is an equation containing an unknown function and its derivatives.

Examples:.

1. $$\frac{dy}{dx} = 2x + 3$$

2. $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + ay = 0$$

3. $$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$$

***y*** is dependent variable and ***x*** is independent variable, and these are ordinary differential equations

# Partial Differential Equation

Examples:

1. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

***u*** is dependent variable and ***x and y*** are independent variables, and is partial differential equation.

2. $$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial t^4} = 0$$

3. $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}$$

***u*** is dependent variable and ***x and t*** are independent variables

# Order of Differential Equation

**The order of the differential equation is order of the highest derivative in the differential equation.**

| **Differential Equation** | **ORDER** |
|---|---|
| $\frac{dy}{dx} = 2x + 3$ | 1 |
| $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 9y = 0$ | 2 |
| $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$ | 3 |

# Degree of Differential Equation

**The degree of a differential equation is power of the highest order derivative term in the differential equation.**

| Differential Equation | Degree |
|---|---|
| $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + ay = 0$ | 1 |
| $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$ | 1 |
| $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + 3 = 0$ | 3 |

# Linear Differential Equation

**A differential equation is linear, if**

1. **dependent variable and its derivatives are of degree one,**
2. **coefficients of a term does not depend upon dependent variable.**

**Example:** 1. $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 9y = 0.$$

**is linear.**

**Example:** 2. $$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$$

**is non - linear because in 2nd term is not of degree one.**

**Example:** 3. $$x^2 \frac{d^2 y}{dx^2} + y \frac{dy}{dx} = x^3$$

**is non - linear because in 2nd term coefficient depends on *y*.**

**Example:** 4. $$\frac{dy}{dx} = \sin y$$

**is non - linear because** $\sin y = y - \frac{y^3}{3!} + -$ is non – linear

## 9. Table 1. Classify each differential equation

| No | Differential Equations | Ordinary or Partial | Linear or nonlinear | Order | Degree | Independent variables | Dependent variables |
|---|---|---|---|---|---|---|---|
| 1. | $y' = x + 6y$ | | | | | | |
| 2. | $y'' = 4y + y^3$ | | | | | | |
| 3. | $\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} - 2y = x^3$ | | | | | | |
| 4. | $y'' + 2xy' + 4y = \cos 2x$ | | | | | | |
| 5. | $\frac{dy}{dx} = \frac{x^2 - 1}{y + 4}$ | | | | | | |
| 6. | $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$ | | | | | | |
| 7. | $\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial t^2}$ | | | | | | |

It is Ordinary/partial Differential equation of order… and of degree…, it is linear / non linear, with independent variable…, and dependent variable….

# 1st – order differential equation

**1.** Derivative form:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

**2.** Differential form:

$$(1 + x)dy - ydx = 0$$

**3.** General form:

$$\frac{dy}{dx} = f(x, y)$$

or

$$f(x, y, \frac{dy}{dx}) = 0.$$

# First Order Ordinary Differential equation

$$f(x, y, \frac{dy}{dx}) = O.$$

| | |
|---|---|
| $\frac{dy}{dx} = f(x, y)$ | Derivative form |
| $M(x, y)dx + N(x, y)dy = 0$ | Differential form |
| $a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$ | Standard form |
| $a_1(x)y' + a_0(x)y = g(x)$ | Standard form |
| $\frac{dy}{dx} + P(x)y = Q(x)$ | First order linear differential equation form |

# Second order Ordinary Differential Equation

$$f(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = O.$$

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx})$$

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_2(x)y\acute{} + a_1(x)y\acute{} + a_0(x)y = g(x)$$

# nth – order linear differential equation

**1.** nth – order linear differential equation with constant coefficients.

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

**2.** nth – order linear differential equation with variable coefficients

$$a_n(x)\frac{dy}{dx} + a_{n-1}(x)\frac{d^{n-1} y}{dx^n} + \dots\dots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

# Solution of Differential Equation

Examples

**$y=3x+c$ , is solution of the 1st order differential equation $\frac{dy}{dx}=3$, $c_1$ is arbitrary constant.**

**As is solution of the differential equation for every value of $c_1$, hence it is known as <u>general solution.</u>**

Examples

$$y' = \sin(x) \Rightarrow y = -\cos(x) + C$$

$$y'' = 6x + e^x \Rightarrow y' = 3x^2 + e^x + C_1 \Rightarrow y = x^3 + e^x + C_1 x + C_2$$

Observe that the set of solutions to the above 1st order equation has 1 parameter,

while the solutions to the above 2nd order equation depend on two parameters.

# Families of Solutions

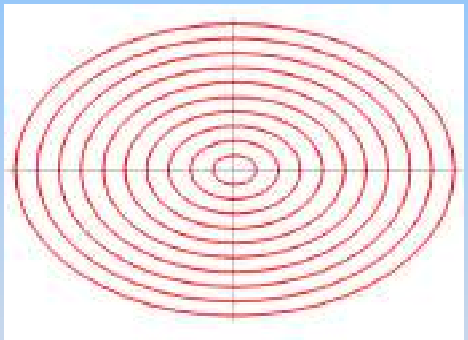
**Example** $9yy' + 4x = 0$

**Solution**

$$\int (9yy' + 4x)\,dx = C_1 \Rightarrow \int 9y(x)y'(x)\,dx + \int 4x\,dx = C_1$$

$$\Rightarrow \int 9y\,dy + 2x^2 = C_1 \Rightarrow \frac{9y^2}{2} + 2x^2 = C_1 \Rightarrow 9y^2 + 4x^2 = 2C_1$$

This yields $\frac{y^2}{4} + \frac{x^2}{9} = C$ where $C = \frac{C_1}{18}$.



Observe that given any point $(x_0, y_0)$, there is a unique solution curve of the above equation which curve goes through the given point.

The solution is a family of ellipses.

# Origin of Differential Equations Solution

## 1. Geometric Origin

1. For the family of straight lines

$$y = c_1 x + c_2$$ the differential equation is

$$\frac{d^2 y}{dx^2} = 0$$

2. For the family of curves

A. $$y = ce^{\frac{x^2}{2}}$$ the differential equation is $$\frac{dy}{dx} = xy$$

B. $$y = c_1 e^{2x} + c_2 e^{-3x}$$

the differential equation is

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

# Physical Origin

**1. Free falling stone**

$$\frac{d^2 s}{dt^2} = -g$$

where s is distance or height and
g is acceleration due to gravity.

**2. Spring vertical displacement**

$$m\frac{d^2 y}{dt^2} = -ky$$

where y is displacement,
m is mass and
k is spring constant

**3. RLC – circuit, Kirchoff 's Second Law**

$$L\frac{d^2 q}{dt^2} + R\frac{dq}{dt} + \frac{1}{c}q = E$$

q is charge on capacitor,
L is inductance,
c is capacitance.
R is resistance and
E is voltage

# Physical Origin

**1.Newton's Low of Cooling**

$$\frac{dT}{dt} = \kappa\left(T - T_s\right)$$

where $\frac{dT}{dt}$ is rate of cooling of the liquid,

$T - T_s$ is temperature difference between the liquid 'T' and its surrounding Ts

**2. Growth and Decay**

$$\frac{dy}{dt} = \kappa\ y$$

y is the quantity present at any time