Fourier Series

Introduction:

Fourier series introduced in 1807 by Jean-Baptiste Joseph Fourier (1768-1830) (after work by Euler and Daniel Bernoulli) was one of the most important developments in applied mathematics. It is very useful in the study of heat conduction, mechanics, concentrations of chemicals and pollutants, electrostatics, acoustics and in areas unheard of in Fourier’s day such as computing and CAT scan(Computer Assisted Tomography).

Fourier series is an infinite series representation of periodic function in terms of the trigonometric sine and cosine functions. Fourier series is very powerful method to solve ordinary and partial differential equations particularly with periodic functions appearing as non-homogeneous terms. While Taylor’s series expansion is valid only for functions which are continuous and differentiable, Fourier series is possible not only for continuous functions but for periodic functions, functions discontinuous in their values and derivatives. Further, because of the periodic nature, Fourier series constructed for one period is valid for all values.

- Dirichlet's conditions:
- Consider a real valued function f(x) satisfying the following conditions called Dirichlet's conditions
- i. f(x) is defined in the interval (a,a+2l) with the property f(x+2l)=f(x) i. e, is periodic with period 2l.
- ii. f(x)is continuous or should have simple discontinuity in the interval (a,a+2l) .
- iii.f(x) has no or only a finite number of maxima or minima in the interval (a,a+2l).

Let $a_0 = \frac{1}{l}\int_a^{a+2l} f(x)\, dx \longrightarrow$ (1)

$$a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\frac{n\pi x}{l}\, dx \quad ,n=1,2,3,\ldots\ldots\infty \longrightarrow$$

(2)

$$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\frac{n\pi x}{l}\, dx \quad ,n=1,2,3,\ldots\ldots\infty \longrightarrow$$

(3)

T

The sum of the infinite series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ is called the trigonometric Fourier series or simply Fourier series converges to the function $f(x)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

The coefficients $a_0, a_n \, and \, b_n$ are called Fourier coefficients. The integrals defined equation (1), (2) and (3) are called Euler's formulae.

Result I: Leibnitz's rule:

$\int u \,.v\, dx = uv_1 - u' v_2 + u'' v_3 - u''' v_4 + \ldots\ldots\ldots\ldots$ where 'denotes differentiation and suffix integration w. r. t x

Result II: $\cos n\pi = (-1)^n$; $\sin n\pi = 0$; $\cos(2n+1)\frac{\pi}{2} = 0$; $\sin(2n+1)\frac{\pi}{2} = (-1)^n$.

Particular Cases:

Case i: Let $a = 0$ then $f(x)$ is defined in the interval $(0, 2l)$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\, dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\, dx \quad , n=1,2,3, \ldots\ldots\infty$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\, dx \quad , n=1,2,3, \ldots\ldots\infty$$

Case ii: Let $a = -l$ then $f(x)$ is defined in the interval $(-l, l)$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\, dx \quad , \text{n=1,2 ,3, ...}\infty$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\, dx \quad , \text{n=1,2 ,3, ...}\infty$$

Case iii: Let $a = 0$, $l = \pi$ then $f(x)$ is defined in the interval $(0, 2\pi)$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x) \cos nx\, dx \quad , \text{n=1,2 ,3, ...}\infty$$

$$b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x) \sin nx\, dx \quad , \text{n=1,2 ,3, ...}\infty$$

Case iv: Let $a = -\pi, l = \pi$ then $f(x)$ is defined in the interval $(-\pi, \pi)$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \quad , n=1,2,3, \ldots\ldots\infty$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \quad , n=1,2,3, \ldots\ldots\infty$$

<u>To derive Euler's Formulae (1), (2) *and* (3):</u>

Remember the values of the following integrals

For any integers p and q when $p \neq q$

1) $\int_a^{a+2l} 1\, dx = 2l$ → (4)
2) $\int_a^{a+2l} \cos\frac{p\pi x}{l}\, dx = 0 = \int_a^{a+2l} \sin\frac{p\pi x}{l}\, dx$ → (5)

1) $\int_{a}^{a+2l} \cos\frac{p\pi x}{l} \ \sin\frac{q\pi x}{l} \ dx = 0 \longrightarrow$ (6)

2) $\int_{a}^{a+2l} \cos\frac{p\pi x}{l} \ \cos\frac{q\pi x}{l} \ dx = 0 \longrightarrow$ (7)

3) $\int_{a}^{a+2l} \sin\frac{p\pi x}{l} \ \sin\frac{q\pi x}{l} \ dx = 0 \longrightarrow$ (8)

If $p = q$

4) $\int_{a}^{a+2l} cos^2\left(\frac{p\pi x}{l}\right) \ dx = l = \int_{a}^{a+2l} sin^2\left(\frac{p\pi x}{l}\right) \ dx \longrightarrow$ (9)

Derivation of Euler's Formulae:

Consider $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \longrightarrow$ (10)

Finding a_0:

$\int_{a}^{a+2l} f(x)\, dx = \frac{a_0}{2}\int_{a}^{a+2l} dx + \sum_{n=1}^{\infty} a_n \int_{a}^{a+2l} \cos\frac{n\pi x}{l} \ dx +$
$\sum_{n=1}^{\infty} b_n \int_{a}^{a+2l} \sin\frac{n\pi x}{l} \ dx$

($\because$ on integration on both sides of equation (10) from a to $a + 2l$)

$\int_{a}^{a+2l} f(x)\, dx = \frac{a_0}{2} \ 2l + 0 + 0$ ($\because$ *using* (4) *and* (5))

$\int_{a}^{a+2l} f(x)\, dx = a_0 \ l$

$\therefore a_0 = \frac{1}{l}\int_{a}^{a+2l} f(x) \ dx$

Finding a_n:

Multiplying $\cos\frac{p\pi x}{l}$ on both sides of equation (10) and integrating from a to $a+2l$ we get

$$\int_a^{a+2l} f(x)\cos\frac{p\pi x}{l}\,dx$$

$$= \frac{a_0}{2}\int_a^{a+2l}\cos\frac{p\pi x}{l}\,dx + \sum_{n=1}^{\infty} a_n \int_a^{a+2l}\cos\frac{n\pi x}{l}\cos\frac{p\pi x}{l}\,dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_a^{a+2l}\sin\frac{n\pi x}{l}\cos\frac{p\pi x}{l}\,dx$$

$$\int_a^{a+2l} f(x)\cos\frac{p\pi x}{l}\,dx = \frac{a_0}{2}(0) + a_p\int_a^{a+2l} cos^2\frac{p\pi x}{l}\,dx + 0 \text{ (when p=n)}$$

$$\int_a^{a+2l} f(x)\cos\frac{p\pi x}{l}\,dx = a_p l$$

$$a_p = \frac{1}{l}\int_a^{a+2l} f(x)\cos\frac{p\pi x}{l}\,dx$$

Changing p to n we get

$$\therefore a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

Finding b_n:

Multiplying $\sin\frac{p\pi x}{l}$ on both sides of equation (10) and integrating from a to $a+2l$ we get

$$\int_a^{a+2l} f(x)\sin\frac{p\pi x}{l}\,dx$$

$$= \frac{a_0}{2}\int_a^{a+2l} \sin\frac{p\pi x}{l}\,dx + \sum_{n=1}^{\infty} a_n \int_a^{a+2l} \cos\frac{n\pi x}{l}\sin\frac{p\pi x}{l}\,dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_a^{a+2l} \sin\frac{n\pi x}{l}\sin\frac{p\pi x}{l}\,dx$$

$\int_a^{a+2l} f(x)\sin\frac{p\pi x}{l}\,dx = \frac{a_0}{2}(0) + \sum_{n=1}^{\infty} a_n\,(0) + b_p\int_a^{a+2l} sin^2\frac{p\pi x}{l}\,dx$ (when p=n)

$$\int_a^{a+2l} f(x)\sin\frac{p\pi x}{l}\,dx = b_p\int_a^{a+2l} sin^2\frac{p\pi x}{l}\,dx$$

$$\int_a^{a+2l} f(x)\sin\frac{p\pi x}{l}\,dx = b_p l$$

$$b_p = \frac{1}{l}\int_a^{a+2l} f(x)\,sin\,\frac{p\pi x}{l}\,dx$$

Changing p to n we get

$$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Half Range Expansions:

Fourier Cosine and Sine series:

So far we have considered the Fourier series expansion of a function which is periodic defined in an interval $(a, a+2l)$ of length $2l$. Now we consider the procedure to expand a non-periodic function $f(x)$ defined in half of the above interval say $(0, l)$ of length l. Such expansions are known as half range expansions or half range Fourier series. In particular, a half range expansion containing only cosine terms is known as half range Fourier cosine series of $f(x)$ in the interval $(0, l)$. In similar way half range Fourier sine series contains only sine terms.

Note that the given function $f(x)$ is neither periodic nor even nor odd. In order to obtain a Fourier cosine series for $f(x)$ in the interval $(0, l)$, we construct a new function $g(x)$ such that

i. $g(x) = f(x)$ in the interval $(0, l)$.
ii. $g(x)$ is even function in the interval $(-l, l)$ and is periodic with period $2l$.

Such a function $g(x)$ is known as the "even periodic continuation (or extension) of $f(x)$". The Fourier cosine series for $g(x)$ is valid in interval $(-l, l)$ (or infact for all x) is readily obtained as $g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ $\rightarrow$ (1)

Where $a_0 = \frac{2}{l}\int_0^l g(x)\, dx = \frac{2}{l}\int_0^l f(x)\, dx$

$$a_n = \frac{2}{l}\int_0^l g(x)\cos\frac{n\pi x}{l}\, dx \quad \text{or} \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\, dx$$

In similar way, to obtain the Fourier sine series for $f(x)$ in the interval $(0, l)$, define $g(x)$ such that

i. $g(x) = f(x)$ in the interval $(0, l)$.
ii. $g(x)$ is odd function in the interval $(-l, l)$ and is periodic with period $2l$.

Such a function $g(x)$ is known as the “odd periodic continuation of $f(x)$". The Fourier cosine series for $g(x)$ is valid in interval $(-l, l)$ (or infact for all x) is readily obtained as

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \longrightarrow (2)$$

Where $b_n = \frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\, dx$ or $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\, dx$