Numerical Analysis :

Introduction:

Numerical Analysis a rapidly growing subject and has gained much importance in recent years. The limitations of analytical methods in practical applications have led mathematicians to evolve numerical methods. Although the numerical data used in solving practical problems are approximate, more over the methods and processes used to find the derived results are approximate, but the information derived from such approximations are very useful. With the advent of high speed computers and increasing demand for numerical solution to various problems, numerical techniques have become indispensable tools in the hands of engineers and scientists. In this chapter we study finite differences, interpolation, numerical differentiation and numerical integration.

Finite Differences:

Introduction:

The calculus of finite differences deals with the changes that take place in the value of function (dependent variable) due to finite changes in the independent variables. Through this we also study the relations that exist between the values assumed by the function, whenever the independent variables changes by finite jumps whether equal or unequal. On the other hand in infinitesimal calculus, we study those changes of the function which occur when the independent variable changes continuously in a given interval. In this chapter we study the variations in the function when the independent variable changes by equal interval.

Finite differences:

Let y=$f(x)$ be a given function of x and let $y_0, y_1, y_2, \ldots\ldots y_n$ be the values of y corresponding to $x_0, x_1, x_2, \ldots\ldots x_n$, the values of x. The independent variable x is called the argument and the corresponding dependent variable y is called entry. In general the difference between any two consecutive values of x need not be same or equal.

We can write the arguments and entries as below:

x	x_0	x_1	x_2......	x_{n-1}	x_n
y	y_0	y_1	y_2......	y_{n-1}	y_n

f we subtract from each value of y (except y_0) the preceding value of y, we get $y_1 - y_0$, $y_2 - y_1$, $y_3 - y_2$,...... $y_n - y_{n-1}$.

by Δy i.e $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1$, $\Delta y_2 = y_3 - y_2$,......$\Delta y_{n-1} = y_n - y_{n-1}$. Here the symbol Δ denotes an operation called FORWARD DIFFERENCE OPERATOR.

HIGHER ORDER DIFFERENCES:

The second and higher differences are defined by

$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$\Delta^2 y_1 = \Delta(\Delta y_1) = \Delta(y_2 - y_1) = \Delta y_2 - \Delta y_1 = (y_3 - y_2)$

Similarly $\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = y_3 - 3y_2 + 3y_1 - y_0$.

We observe that the coefficients occurring in RHS are simply the binomial coefficients in $(1-x)^n$. $Hence$ in general we have $\Delta^n y_0 = y_n - nc_1 y_{n-1} + nc_2 y_{n-2} + \ldots\ldots + (-1)^n y_0$.

Also $\Delta^n y_i = \Delta^{\mathrm{n-1}} y_{i+1} - \Delta^{\mathrm{n-1}} y_i$.

Though the arguments $x_0,\ x_1,\ x_2, \ldots\ldots.$need not in general be equally spaced for purpose of practical work, we take them equally spaced.

Usually the arguments are taken as

$x_0, x_0 + h, x_0 + 2h, x_0 + 3h, \ldots\ldots.$

So that $x_1 - x_0 = x_2 - x_1, x_3 - x_2 = \ldots\ldots. = h, h$ is called the interval of differencing.

The first forward difference of y=f(x) is defined as Δf(x)=f(x+h)-f(x).

The second forward difference of y=f(x) is defined as

Δ^2f(x)= Δ{Δf(x)}

= Δ[f(x+h)-f(x)]

= Δf(x+h)- Δf(x)

= [f(x+2h)-f(x+h)]- [f(x+h)-f(x)]

= f(x+2h)-2f(x+h)+f(x)

The general n[th] forward difference is defined

Δ^{n}f(x)= Δ^{n-1}f(x+h)- Δ^{n-1}f(x)

These differences are systematically set out as follows in the form of a forward difference table.

Forward difference table:

Argument (x)	Entry y	First difference	Second Difference	Third difference	Forth Difference
x_0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
$x_1 = x_0 + h$	y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	
$x_2 = x_0 + 2h$	y_2	Δy_2	$\Delta^2 y_2$		
$x_3 = x_0 + 3h$	y_3	Δy_3			
$x_4 = x_0 + 4h$	y_4				

Note: In this table y_0 is known as the 1st entry is called the leading term and $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0$ are called the leading differences.

Properties of the operator Δ:

1. Δ is linear i.e $\Delta[af(x) + bg(x)] = a\Delta f(x) + b\Delta g(x)$
   Where a and b are constants.
   Proof: $\Delta[af(x)+bg(x)]$
   $= [af(x+h)+bg(x+h)] - [af(x)+bg(x)]$
   $= a[f(x+h)-f(x)] + b[g(x+h)-g(x)]$
   $= a\Delta f(x) + b\Delta g(x)$
2. $\Delta^m \Delta^n[f(x)] = \Delta^{m+n}[f(x)]$
   Proof: $\Delta^m \Delta^n[f(x)]$
   = [Δ.Δ.Δ.....m times][Δ.Δ.Δ...n times]f(x)
   =[Δ.Δ.Δ.....m+n times]f(x)
   $= \Delta^{m+n}[f(x)]$

1. $\Delta[f(x)g(x)]=f(x+h)\,\Delta[g(x)]+g(x)\,\Delta[f(x)]$

Proof: $\Delta[f(x)g(x)]$

$= f(x+h)g(x+h)- f(x)g(x)$

$[f(x+h)g(x+h)- f(x+h)g(x)]+[f(x+h)g(x)-f(x)g(x)]$

$=f(x+h)[g(x+h)-g(x)]+g(x)[f(x+h)-f(x)]$

$= f(x+h)\,\Delta[g(x)]+ g(x)\,\Delta[f(x)]$

2. $\Delta\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\Delta[f(x)]-f(x)\Delta[g(x)]}{g(x+h)\,g(x)}$

Proof: $\Delta\left[\frac{f(x)}{g(x)}\right]$

$= \frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}$

$=\frac{f(x+h)g(x)-g(x+h)f(x)}{g(x+h)\,g(x)}$

$=\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{g(x+h)\,g(x)}$

$=\frac{g(x)[f(x+h)-f(x)]-f(x)[g(x+h)-g(x)]}{g(x+h)\,g(x)}$

$=\frac{g(x)\Delta[f(x)]-f(x)\Delta[g(x)]}{g(x+h)\,g(x)}$

1. If c is a constant then $\Delta c=0$

   Proof: Let $f(x)=c$

   Therefore $f(x+h)=c$ (where h is the interval of differencing)

   $\Delta f(x)=f(x+h)-f(x)=c-c=0$

   $\Rightarrow \Delta c=0$

   Note :If n is a positive integer $\Delta^{n}[\Delta^{-n}f(x)]=f(x)$ and in particular when n=1 $\Delta[\Delta^{-1}f(x)]=f(x)$.

Fundamental theorem for finite differences:

Theorem: the n^{th} differences of a polynomial of the n^{th} degree are Constant and all higher order differences are zero i.e $\triangle^r[f(x)] = \{$constant if r=n

o if r>n

Proof: let f(x) $= a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots. + a_{n-1} x + a_n$ where a_o≠o be a polynomial of degree n. (a_o is called the leading coefficient of f(x))

$\triangle$f(x)=f(x+h)-f(x)

$= [a_o(x+h)^n + a_1(x+h)^{n-1} + \ldots\ldots. + a_{n-1}(x+h) + a_n] - [a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_{n-1} x + a_n$

$= a_0[(x+h)^n - x^n] + a_1[(x+h)^{n-1} - x^{n-1}] + \ldots\ldots + a_{n-1}[x+h-x]$

$= a_0[x^n + nc_1 x^{n-1} h + nc_2 x^{n-2} h^2 + \ldots.. + h^n - x^n] + a_1[x^{n-1} + n-1 c_1\, x^{n-2} h + \ldots.. + h^{n-1} - x^{n-1}] + \ldots. + a_{n-1} h$

= $a_0 n h x^{n-1} + [a_0 h^2 nc_2 + a_1 h$(n-1)]$x^{n-2}$ + …. $+ a_{n-1} h$

$$=a_0 n h x^{n-1} + [a_o h^2 nc_2 + a_1 h(n-1)] x^{n-2} + \ldots. + a_{n-1} h$$

$$=h x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \ldots.. + b_{n-1} x + b_n \ldots\ldots\ldots(1)$$

Where b_2,b_3……b_n are constants

From (1) it is clear that the first difference of f(x) is a polynomial of $(n-1)^{th} degree$.

Similarly $\Delta^2 f(x) = \Delta[\Delta f(x)] = \Delta[f(x+h) - f(x)] = \Delta f(x+h) - \Delta f(x)$

=$a_0nh[(x+h)^{n-1} - x^{n-1}] + b_2[x+h)^{n-2} - x^{n-2}] + \ldots\ldots + b_{n-1}[x+h-x]$

$$a_0n(n-1)h^2x^{n-2} + c_3x^{n-3} + c_4x^{n-4} + \ldots.. + c_{n-1}x + c_n$$

Where $c_3, c_4 \ldots\ldots c_{n-1}$ $are\, constants$.

Therefore the second differences of f(x) reduces to a polynomial of

$(n-2)^{th}$ degree.

Proceeding as above and differencing for a times we get

$$\Delta^n f(x) = a_0[n(n-1)\ldots..3*2*1]\, h^n x^{n-n}$$

$$= a_0 n! h^n$$

Which is a constant and $\Delta^{n+1} f(x) = \Delta[\Delta^n f(x)] = \Delta[a_0 n!\, h^n] = 0$ (because $\Delta c = o$)

which completes the proof of the theorem.

NOTE: The converse of the above theorem is true i.e if the n^{th} differences of a function tabulated at equally spaced intervals are constant, the function is polynomial degree of n.

Example: Evaluate $\Delta^3(1-x)(1-2x)(1-3x)$

Solution:-

Let f(x) = (1-x)(1-2x)(1-3x)=$-6x^3+11x^2-6x+1$

f(x) is a polynomial of degree 3 and the coefficient of x^3 is (-6)

Here $a_n=-6,\ h=1,\ n=3 \quad \therefore \Delta^3 f(x)=(-6)3!=-36$

Applications of finite differences:

1.It is mainly applicable in cloud computing, data storage, IT security and Big data are EMC products solutions that enable other business to store, manage, protect and analyse their data securely.

2. The biological applications are involved in Numerically to studyof

water diffusion in biological tissues using an improved finite difference.

An improved finite difference (FD) has been developed in order to calculate the behaviour of the nuclear magnetic resonance signal variations caused by water diffusion in biological tissues more accurately and efficiently.

2.In Journal of Computational and applied mathematics finite difference method is very useful to solve the forward problem in electroencephalography (EEG).

3.In physical interest, Non standard finite differences can be used to construct exact algorithms to solve some differential equations such as the wave equation and Schrodinger's equation.

Even where exact algorithm do not exist, non standard finite differences can greatly improve the accuracy of low-order finite-difference algorithms.