PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations are those equations which contain partial derivatives,independent Variables and dependent variables.

The independent variables will be denoted by ‘x’ & ‘y’ and dependent variable by ‘z’.

The Partial differential coefficients are denoted as follows

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial y^2} = t, \frac{\partial^2 z}{\partial x \partial y} = s$$

ORDER:

Order of partial differential equations is the same as that of the highest differential coefficients in it.

METHODS OF FORMING PARTIAL DIFFERENTIAL EQUATION

The PDE'S formedby 2 methods

1) By eliminating arbitrary constants
2) By eliminating arbitrary functions

Methods of eliminating arbitrary constants

Ex;-form of PDE from $x^2+y^2+(z-c)^2=a^2$

$X^2+y^2+(z-c)^2=a^2$ ....> (1)

The above equation contains 2 arbitrary constants a & c

Differentiate equation (1) partial w.r.t, x we get

$$2x+2(z-c)\frac{\partial z}{\partial x}=0$$

X+ (z-c) p=0 ....> (2)

Differentiate (1) partially w.r.t, ywe get

$$2y+2(z-c)\frac{\partial z}{\partial y}=0$$

Y+ (z-c) q=0 ....> (3)

Eliminating c from (2) and (3)

(2).....> z-c=-x/p

Substituting above value in (3) we get

Y-x/p*q=0

Y*p-x*q=0

Methods of eliminating of arbitrary functions

Ex;-form of PDE fromz=f(x^2-y^2)

z=f(x^2-y^2) ....> (1)

Differentiate (1) partially w.r.t, x and y we get

p=$\frac{\partial z}{\partial x} = f$'($x^2$-$y^2$) 2x .....> (2)

q=$\frac{\partial z}{\partial y} = f$'($x^2$-$y^2$) (-2y) ......>(3)

Dividing (2) and (3) we get

P/q=-x/y (or) p*y=-q*x

(Y*p)+(x*q)=0

LAGRANGE'S LINEAR EQUATION IS AN EQUATION OF THE TYPE P*p +Q*q= R. where P, Q, R are the functions of x, y, z and $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$

Proof; - P* p + Q* q= R ....> (1)

The form of the equation is obtained by eliminating an arbitrary function f from

F (u, v) = 0........> (2)

Where u, v, w, are the functions of x, y, z.

Differentiating (2) partially w .r. t, x and y

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}*\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}*\frac{\partial z}{\partial x}\right)=0 \quad ....> (3)$$

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}*\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}*\frac{\partial z}{\partial y}\right)=0 \quad ....> (4)$$

Let us eliminate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (3) and (4)

From (3), $\frac{\partial f}{\partial u}(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\ p) = -\frac{\partial f}{\partial v}(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\ p)$ ....> (5)

From (4), $\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right) = -\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\ q\right)$ .... > (6)

Dividing (5) and (6) we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\ p\div\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q=\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\ P\div\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\ q$$

$$(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\ P)\ \left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\ q\right)=\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\ q\right)(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\ p)$$

$$\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial z}\ q+\frac{\partial u}{\partial z}\times p\times\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial z}\ pq$$
$$=\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial z}\ p+\frac{\partial u}{\partial z}\ q\times\frac{\partial v}{\partial x}+\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial z}\ pq$$

$$\left(\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x}\quad ....>$$
(7)

If (1) and (7) are the same then coefficients p, q is equal

$$P=\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial z}$$

$$Q=\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial z}$$

$$R=\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x}$$

Now suppose u=c1 and v=c2 are 2 solutions where c1 and c2 are constants

Differentiating u=c1 and v=c2

$\frac{\partial u}{\partial x}\ dx + \frac{\partial u}{\partial y}\ dy + \frac{\partial u}{\partial z}\ \ dz = 0$ ....> (9)

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z}\ dz = 0 ...... > (10)$$

Solving (9) and (10) we get

$dx \div \frac{\partial u}{\partial y} \times \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \times \frac{\partial v}{\partial y} = dy \div \frac{\partial u}{\partial z} \times \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \times \frac{\partial v}{\partial z} = dz \div \frac{\partial u}{\partial x} \times \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \times \frac{\partial v}{\partial x}$
........> (11)

From (8) and (10) we get

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Solutions of these equations are u=c1 and v=c2

Therefore f(u, v)= 0 is the required solution of (1)

METHOD OF MULTIPLIERS

Let the auxiliary equation be $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

L, m, n may constants (or) functions of x, y, z then we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = ldx + mdy + ndz \div lP + mQ + nR$$

L, m, n are chosen in such a way that

L*P+ m*Q+ n*R=0

Thus l dx+ m dy +n dz= 0

Solve this differential equation, if the solution is u=c1

Similarly choose another set of multipliers (l1, m1, n1)

And if the second solution is v=c2

Therefore required solution is f (u, v) = 0

Example; - solve $(mz - ny)\frac{\partial z}{\partial x} + (nx - ly)\frac{\partial z}{\partial y} = ly - mx$

Here the auxiliary equations are

$$dx \div mx - ny = dy \div nx - lz = dz \div ly - mx$$

Using the multipliers x, y, z we get

Each fraction $= x\,dx + y\,dy + z\,dz \div x(mz - ny) + (y(nx - lz) + z(ly - mx)$

$$= x\,dz + y\,dy + z\,dz \div 0$$

$$x\,dx + y\,dy + z\,dz = 0$$

0n integration

$X^2+y^2+z^2= c1$ …….> (1)

Again using multipliers l, m, n we get

Each fraction $= l\,dx + m\,dy + n\,dz \div l(mz - ny) + m(nx - lz) + n(ly - mx)$

$$= l\,dx + m\,dy + n\,dz \div 0$$

$$l\,dx + m\,dy + n\,dz = 0$$

On integration

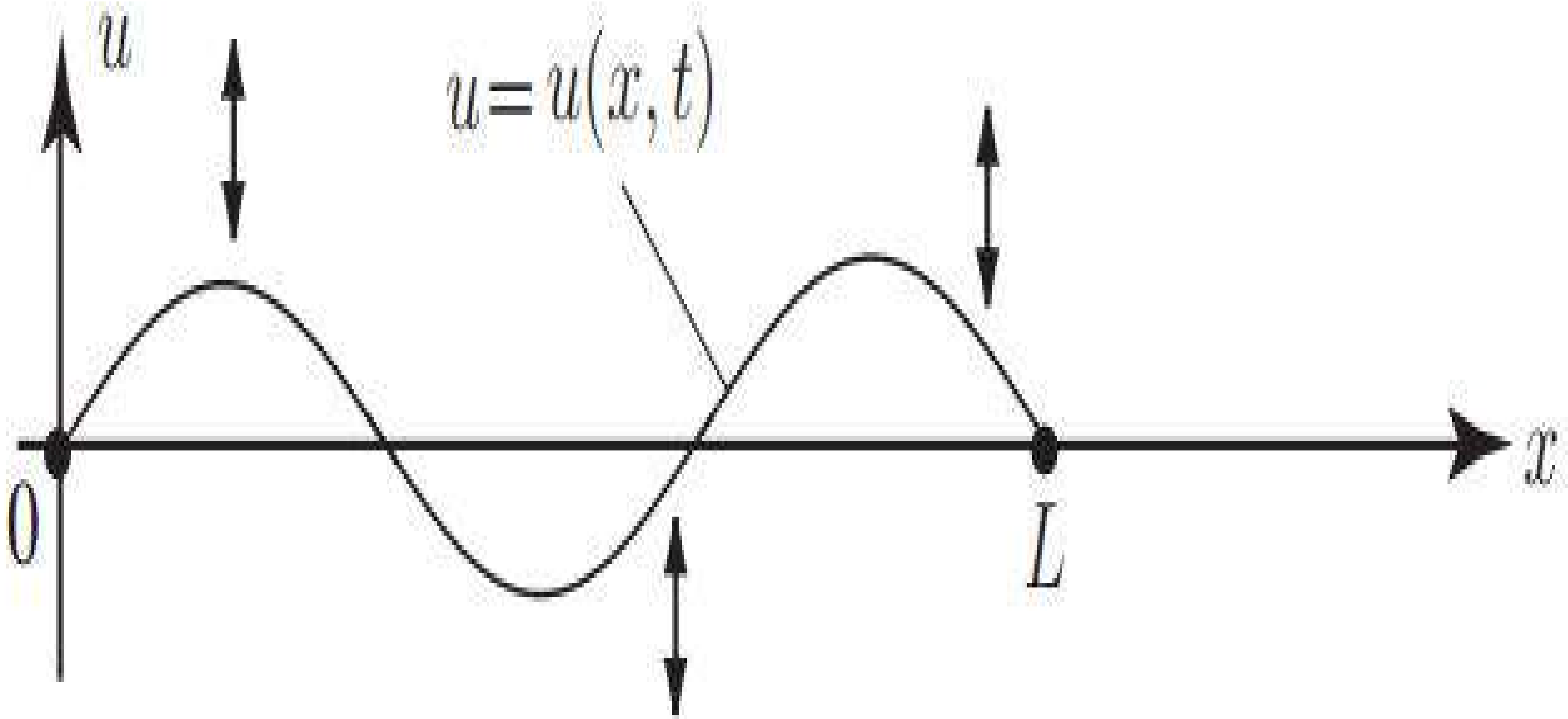
L x +m y+ n z= c2

Hence from (1) and (2) the required solution is $f(x^2+y^2+z^2, lx+my+nz)=0$

APPLICATIONS OF PDEs:

1. WAVE EQUATIONS

The simplest situation to give rise to the one-dimensional wave equation is the motion of a Stretched string - specifically the transverse vibrations of a string such as the string of a Musical instrument. Assume that a string is placed along the x−axis, is stretched and then Fixed at ends x=0andx=L it is then deflected and at some instant, which we call t=0, is Released and allowed to vibrate. The quantity of interest is the deflection u of the string at any Point x, $0 \le x \le L$, and at any time $t>0$. We write u=u(x t). The diagram shows a possible Displacement of the string at a fixed time t.



u
u = u(x, t)
0
L
x

Subject to various assumptions⋯

1. Neglecting damping forces such as air resistance
2. Neglecting the weight of the string
3. That the tension in the string is tangential to the curve of the string at any point
4. That the string performs small transverse oscillations i.e. every particle of the string moves

Strictly vertically and such that its deflection and slope as every point on the string is small.

⋯ it can be shown, by applying Newton's Law of motion to a small segment of the string, that u satisfies the PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $c^2 = \frac{T}{\rho}$, ρ being the mass per unit length of the string and T being the (constant) horizontal component of the tension in the string. To determine $u(x,t)$ uniquely, we must also know

1. The initial definition of the string at time t=0 at which it is released
2.The initial velocity of the string. Thus we must be given initial conditions

$$u\ (x,0) = f(x) \qquad 0 \le x \le L \qquad \text{(Initial position)}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \qquad 0 \le x \le L \qquad \text{(Initial velocity)}$$

Where f(x) and g(x) are known. These two initial conditions are in addition to the two boundary conditions

U (0, t) =u (L, t) =0 for t≥0 which indicate that the string is fixed at each end. In the specific example discussed in the previous Section we had

$$f(x) = u_0 \ \sin\left(\frac{\pi x}{\ell}\right)$$

$$g(x) = 0 \qquad \text{(string initially at rest).}$$

The PDE (1) is the (undamped) wave equation. We will discuss solutions of it for various initial conditions later. More complicated forms of the wave equation would arise if some of the assumptions were modified. For example:

(a) $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} - g$ if the weight of the string was allowed for,

(b) $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} - \alpha \dfrac{\partial u}{\partial t}$ if a damping force proportional to the velocity of the string (with damping constant α) was included.

This is referred to as the one-dimensions l wave equation because only one space variable,x,is present. The two-dimensional (undamped) wave equation is, in Cartesian coordinates,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

This arises for example when we model the transverse vibrations of a membrane. See Figures below. Here u(x, y, t)is the definition of a point (x, y)on the membrane at time t. Again, a boundary condition must be specified: commonly

u=0 t≥0 on the boundary of the membrane, if this is fixed (clamped). Also initial condition must given

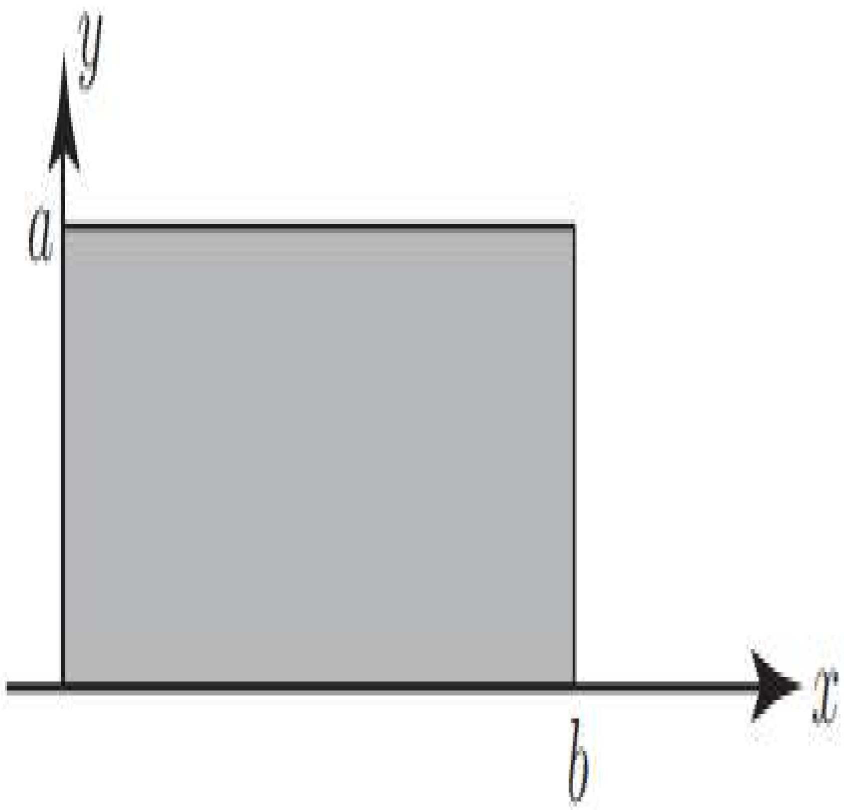
$$u\ (x, y, 0) = f(x, y) \qquad \text{(initial position)}$$

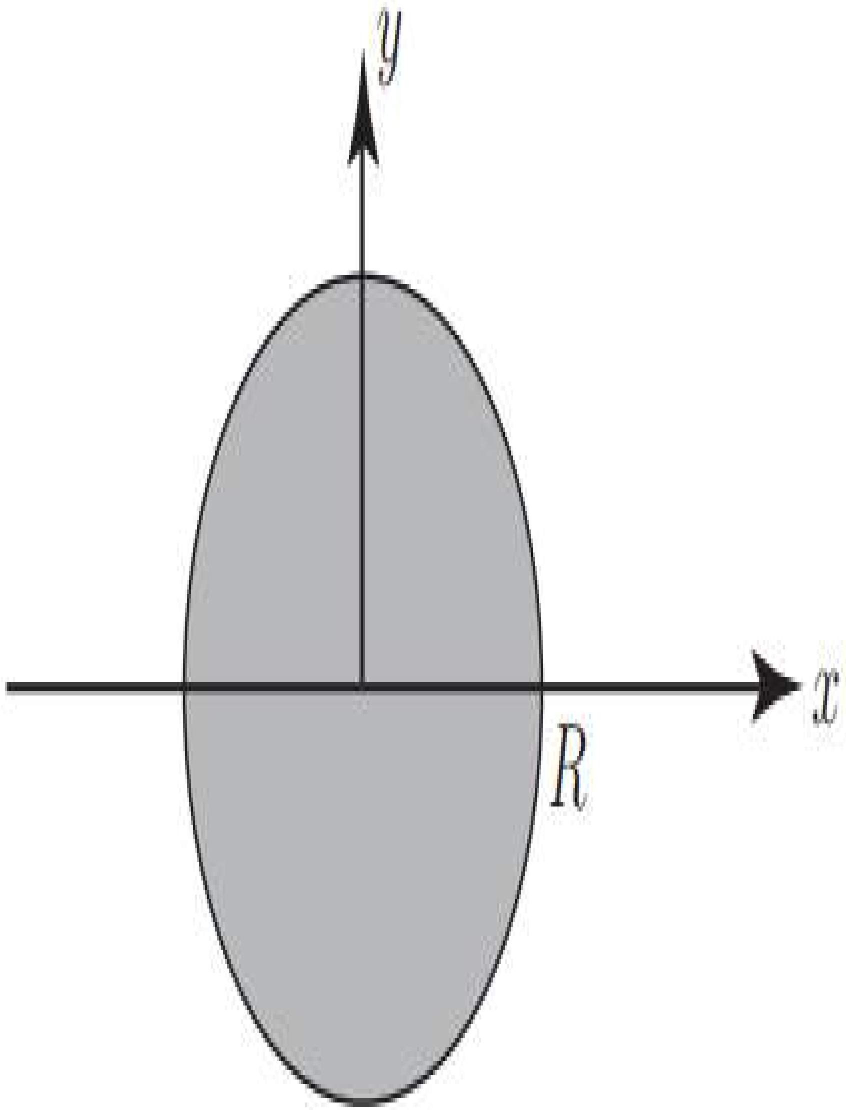
$$\frac{\partial u}{\partial t}(x, y, 0) = g(x, y) \qquad \text{(initial velocity)}$$

Circular membrane, such as a drumhead, polar coordinates defined by x=r cosθ, y=Rsinθ would be more convenient than Cartesian. In this case (2) becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right) \qquad 0 \le r \le R, \qquad 0 \le \theta \le 2\pi$$

for a circular membrane of radius R.


y
a
x
b


y
x
R

2. HEAT CONDUCTION EQUATIONS

Consider a long thin bar, or wire, of constant cross-section and of homogeneous material oriented along the x–axis.

Imagine that the bar is thermally insulated laterally and is sufficiently thin that heat flows (by conduction) only in the x–direction. Then the temperature u at any point in the bar depend only on the x–coordinate of the point and the time t.By applying the principle of conservation of energy it can be shown that u(x, t)satisfies the PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \begin{array}{c} 0 \le x \le L \\ t > 0 \end{array}$$

Where k is a positive constant. In fact k, sometimes called the thermal diffusivity of the bar, is given by

$$k = \frac{\kappa}{s\rho}$$

Where κ=thermal conductivity of the material of the bar

s=specific heat capacity of the material of the bar

ρ=density of the material of the bar.

The PDE (3) is called the one-dimensional heat conduction equation (or, in other contexts where

It arises, the diffusion equation).

- Both equations involve second derivatives in the space variable x but whereas the wave equation has a second derivative in the time variable to the heat conduction equation has only a first derivative int. This means that the solutions of (3) are quite different in form from those of (1) and we shall study them separately later. The fact that (3) is first order in the means that only one initial condition at t =0 is needed, together with two boundary conditions, to obtain a unique solution. The usual initial condition specifies the initial temperature distribution in the bar $u(x,0) = f(x)$ Where f(x)is known. Various types of boundary conditions at x=0and x=L are possible. For example:

(a) $u(0,t) = T_1$ and $u(L,T) = T_2$ (ends of the bar are at constant temperatures T_1 and T_2).

(b) $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$ which are **insulation** conditions since they tell us that there is no heat flow through the ends of the bar.

As you would expect, there are two-dimensional and three-dimensional forms of the heat conduction equation. The two dimensional version of (3) is

$$\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Where u(x, y, t) is the temperature in a flat plate. The plate is assumed to be thin and insulated on its top and bottom surface so that no heat flow occurs other than in the O xy plane. Boundary conditions and an initial condition are needed to give unique solutions of (4). For example if the plate is rectangular:

typical boundary conditions might be

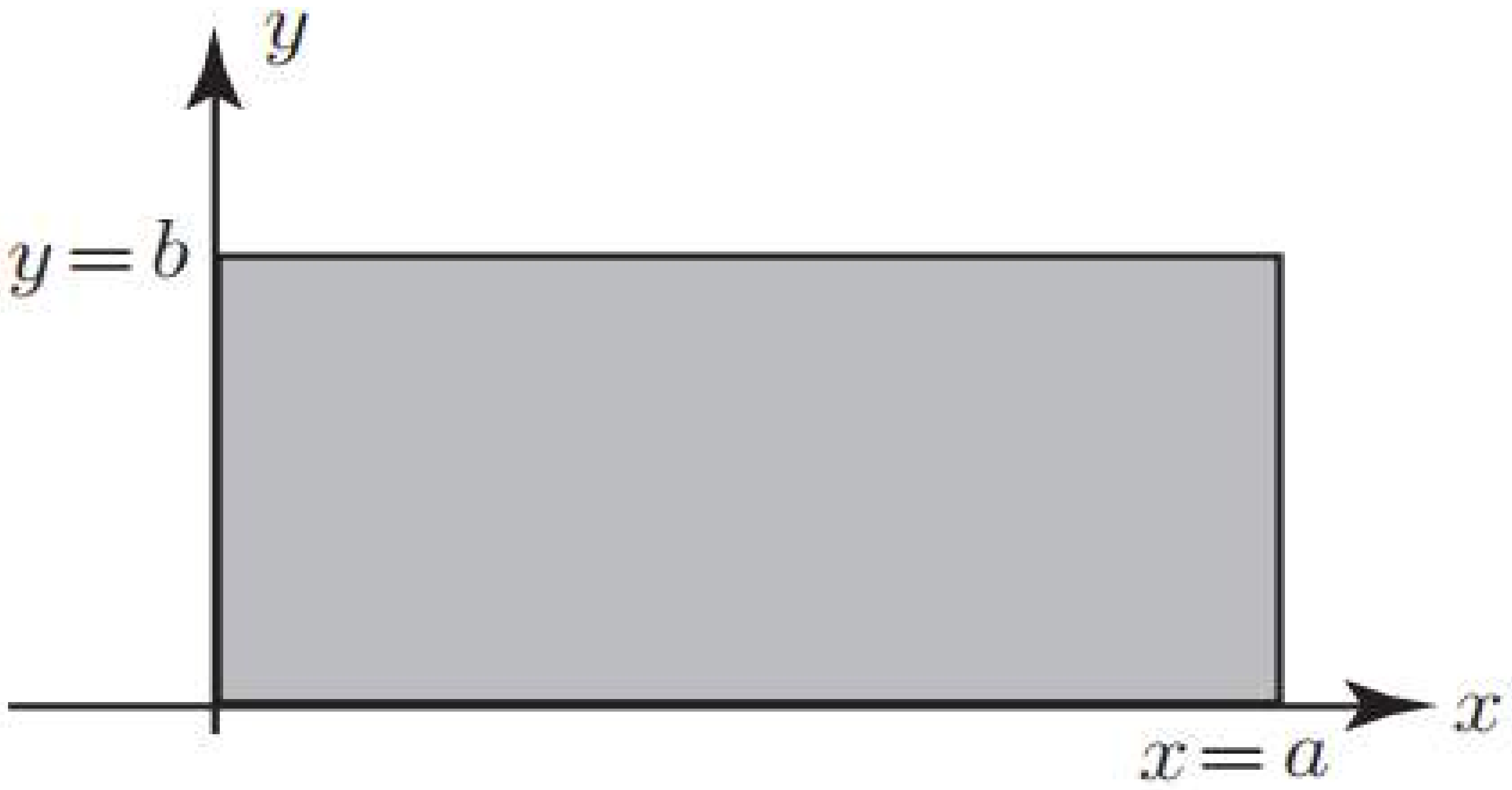
$$u(x,0) = T_1 \qquad 0 \leq x \leq a \text{ (bottom side at fixed temperature)}$$

$$\frac{\partial u}{\partial x}(a,y) = 0 \qquad 0 \leq y \leq b \text{ (right hand side insulated)}$$

$$u(x,b) = T_2 \qquad 0 \leq x \leq a \text{ (top side at fixed temperature)}$$

$$u(0,y) = 0 \qquad 0 \leq y \leq b \text{ (left hand side at zero fixed temperature).}$$

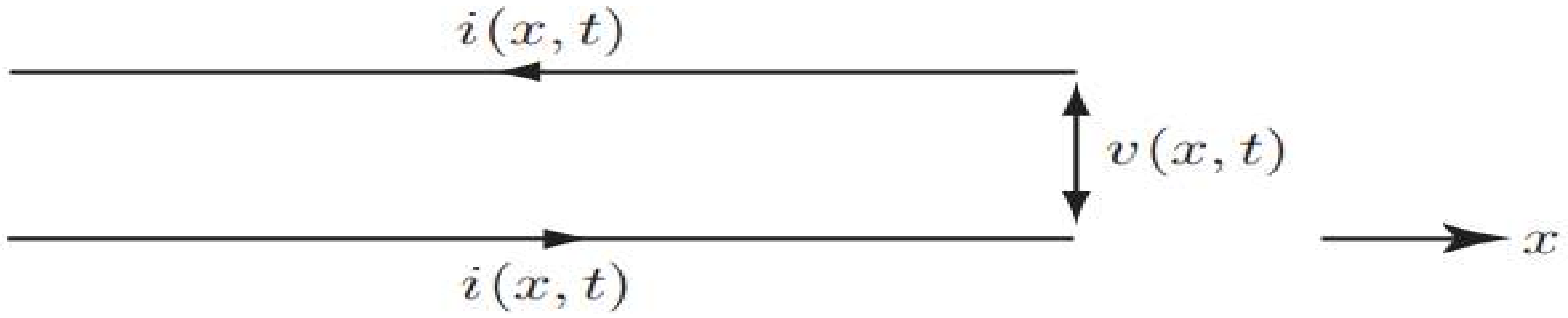
An initial condition would have the form $u(x,y,0) = f(x,y)$, where f is a given function.


y
y = b
x
x = a

3. TRANSMISSION LINE EQUATONS

In a long electrical cable or a telephone wire both the current and voltage depend upon position

Along the wire as well as the time.



It is possible to show, using basic laws of electrical circuit theory, that the electrical current

I(x, t)satisfies the PDE

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (RC + GL)\frac{\partial i}{\partial t} + RGi$$

Where the constants R, L, C and G are, for unit length of cable, respectively the resistance, inductance, capacitance and leakage conductance. The voltage v(x, t) also satisfies (5). Special cases of (5) arise in particular situations. For a submarine cable G is negligible and frequencies are low so inductive effects can also be neglected. In this case (5) becomes

$$\frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t}$$

Which is called the submarine equation or telegraph equation For high frequency alternating currents, again with negligible leakage, (5) can be approximated by

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2}$$

Which is called the high frequency line equation

4. LAPLACE TRANSFORMS

If you look back at the two-dimensional heat conduction equation (4)

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

it is clear that if the heat flow is steady, i.e. time independent, then $\frac{\partial u}{\partial t} = 0$ so the temperature $u(x, y)$ is a solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{8}$$

(8) Is the two-dimensional Laplace equation. Both this and its three-dimensional counterpart

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Arise in a wide variety of applications, quite apart from steady state heat conduction theory.

Since time does not arise in (8) or (9) it is evident that Laplace's equation is always a model

For equilibrium situations. In any problem involving Laplace's equation we are interested in

Solving it in a specific region R for given boundary conditions. Since conditions may involve

(a) u specified on the boundary curve C (two dimensions) or boundary surface S (three dimensions) of the region R. Such boundary conditions are called **Dirichlet conditions**.

(b) The derivative of u **normal** to the boundary, written $\frac{\partial u}{\partial n}$, specified on C or S. These are referred to as **Neumann boundary conditions**.

(c) A mixture of (a) and (b).

Some areas in which Laplace's equation arises are

(a) Electrostatics (u being the electrostatic potential in a charge free region)

(b) Gravitation (u being the gravitational potential in free space)

(c) Steady state flow of in viscid fluids

(d) Steady state heat conduction (as already discussed.)

Other important PDEs in science and engineering

1. **Poisson's equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \qquad \text{(two-dimensional form)}$$

where $f(x, y)$ is a given function. This equation arises in electrostatics, elasticity theory and elsewhere.

2. Helmholtz's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0 \qquad \text{(two dimensional form)}$$

which arises in wave theory.

3. Schrodinger's equation

$$-\frac{h^2}{8\pi^2 m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)=E\psi$$

which arises in quantum mechanics. (h is Planck's constant.)

4. Transverse vibrations in a homogeneous rod

$$a^2\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial t^2}=0$$

where $u(x,t)$ is the displacement at time t of the cross-section through x.

All the PDEs we have discussed are **second order** (because the highest order derivatives that arise are second order) apart from the last example which is **fourth order**.