# VECTOR DIFFERENTIAL CALCULUS

## INTRODUCTION:

Vector calculus is a branch of mathematics concerned with differential and integration of vector field, primarily in 3-dimensional space R3.

It was developed by J. Willard Gibbs and Heaviside.

## BASIC OBJECTS:

Scalar: A physical quantity which has magnitude only is called as a Scalar. Example: every real number is a scalar

Vector: A physical quantity which has both magnitude and direction is called as a Vector.

Example: Velocity, Acceleration.

## VECTOR POINT FUNCTION:

If to each point P(x, y, z) of a region R in the space , there is associated a unique vector F(P) or F(x,y,z) then F is called a vector point function . The set of all points of the region R together with the set of all values of the function F constitute a vector field over R

Example1 : $\nabla = xi + yj + zk$ is a vector point function, which associates with each point (x, y, z) a vector pointing away from the origin. This represents a three –dimensional source field.

Example 2: in theoretical physics , there is associated with each point in space an electric intensity vector , representing the force that would be exerted per unit charge on a charged particle . if it were located at that point . this electric field at any instant of time , constitute a vector field .

Magnetic fields and gravitational fields also provide examples of vector fields defined in space.

- SCALAR POINT FUNCTION:

- Consider any region R of space and suppose that to each point P(x,y,z) of the region in space there corresponds by any law whatsoever , a scalar denoted by (P) or (x,y,z) . we then say that is a scalar point function over the region R . The points of the region R together with the functional values (p) will form a scalar field over R .

- Example 1: If P = (x, y) then (P) =x2+y2 is a scalar point function and it forms a two dimensional scalar field.
- Example2 : if P=(x,y,z) then x2 + y2 + z2 is a scalar point function and it forms a three dimensional scalar field.
- Example3 : Physical examples of a scalar field are,
- a. The mass density of the atmosphere.
- b. The temperature at each point in an insulated wall.
- c. The water pressure at each point in an ocean

- VECTOR OPERATIONS:

The basic algebraic operations in vector calculus are referred to as vector algebra, being defined for a space and then globally applied to vector field. It consists of,

Scalar multiplications: Multiplication of scalar field and a vector field, yielding a vector field, a v

Vector addition: Addition of two vector fields, yielding a vector field, v1 + v2

Dot product: Multiplication of two vector fields, yielding a scalar fields, v1.v2

Cross product: Multiplications of two vector fields, yielding a vector field, v1×v2.

There are also two triple products:

Scalar triplet product:

The dot product of a vector and a cross product of two vectors:

$v1 . (v2 \times v3)$

$v1 . (v2 \times v3) = v2 . (v3 \times v1) = v3 . (v1 \times v2)$

## Vector triple product

The cross product of a vector and a cross product of two vectors:

$v_1 \times (v_2 \times v_3)$

the following relationship holds :

$v_1 \times (v_2 \times v_3) = v_2 (v_1 . \quad v_3) - v_3(v_1 . v_2)$ triple product expansion .

## DIFFERENTIAL OPERATORS:

Vector calculus studies various differential operators defined on a scalar and a vector fields, which are typically expressed in terms of del operator '$\nabla$ ', also known as ' nabla '

The vector differential operator denoted by $\nabla$ is defined by,

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} \ j + \frac{\partial}{\partial z} \ k = \sum \frac{\partial}{\partial x} \ i$$

Where i, j, k are the unit vectors

Now, we define the following quantities which involve the above operator,

- Gradient of a Scalar field
- Divergence of a Vector field
- Curl of the Vector field

Gradient of a scalar point function:

Definition: let $\varphi$ (x, y, z) be a continuously differentiable scalar point function. The gradient of $\varphi$ is denoted by grad $\varphi$ or $\nabla \varphi$, and it is defined by,

$$\text{Grad } \varphi = \nabla \varphi = \left( \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right)\varphi$$

From the definition of gradient we can see that $\nabla \varphi$ or grad $\varphi$ is vector point function. Thus the gradient of a scalar point function is a vector point function.

Note: Gradient of a constant scalar is the zero vectors.

PROPERTIES:

If ` f ` and ` g ` are the continuous and differentiable scalar point functions then,

- $\nabla (f \pm g) = \nabla f \pm \nabla g$

Proof: we have $\nabla (f \pm g) = \sum i \frac{\partial}{\partial x} (f \pm g)$

$$= (\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k)(f \pm g)$$

$$= \frac{\partial}{\partial x}(f \pm g) i + \frac{\partial}{\partial y}(f \pm g) j + \frac{\partial}{\partial z}(f \pm g) k$$

$$= (\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k) \pm (\frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j + \frac{\partial g}{\partial z} k)$$

$$= \nabla f \pm \nabla g$$

- $\nabla (f/g) = g \nabla f - f \nabla g/g^2$ $(g! = 0)$

Proof: we have $\nabla (f/g) = \sum (i\frac{\partial}{\partial x}) (f/g)$

$= \sum 1/g^2 (g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x}) i$

$= 1/g^2 (g \sum \frac{\partial f}{\partial x} i - f \sum \frac{\partial g}{\partial x} i)$

$= g \nabla f - f \nabla g / g^2$

- $\nabla (f g) = f \nabla g + g \nabla f$

Proof: we have, $\nabla (f g) = \sum (\frac{\partial}{\partial x}) (f g) i$

$= \sum (f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x}) i$

$= f (\sum \frac{\partial g}{\partial x}) i + g (\sum I \frac{\partial f}{\partial x})$

$= f \nabla g + g \nabla f$

- $\nabla(c\, f) = c\, \nabla f$

Proof:

$$\nabla(c\, f) = \sum \frac{\partial}{\partial x}(c\, f)\; i$$

$$= \frac{\partial}{\partial x}(cf)i + \frac{\partial}{\partial y}(cf)\; j + \frac{\partial}{\partial z}(cf)\; k$$

$$= c\left( \frac{\partial}{\partial x} i + \frac{\partial}{\partial y}\; j + \frac{\partial}{\partial z}\; k \right) = c\, \nabla f$$

<u>SOME CONSEQUENCES:</u>

1. Show that $d\varphi = \nabla\varphi\, dr$

Proof: let $r = xi + yj + zk$, then

$$dr = dxi + dyj + dzk$$

If $\varphi$ is any scalar point function, then

$$d\varphi = \frac{\partial \varphi}{\partial x}(dx) + \frac{\partial \varphi}{\partial y}\; (dy) + \frac{\partial \varphi}{\partial z}(dz)$$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y}\; j + \frac{\partial}{\partial z}\; k\right). (i\, dx + j\, dy + k\, dz)$$

$$= \nabla\varphi.\, dr$$

2. Let r = ∣r∣ , where r = xi + yj + zk, then prove that,

a. ∇r = r / r

$$\nabla r = \sum \frac{\partial}{\partial x}(r)\, i$$

Given, r = xi + yj + zk and r = ∣r

Then, r $= \sqrt{x^2 + y^2 + z^2} \Rightarrow r2 = x^2 + y^2 + z^2$

Differentiating above equation partially w.r.t x, we get,

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = x/r \quad \text{-------------- (1)}$$

Similarly, differentiating partially w r t y and z, we get

$$\frac{\partial r}{\partial y} = y/r \quad \text{---------- (2)} \quad \text{and} \quad \frac{\partial r}{\partial z} = z/r \quad \text{--------- (3)}$$

Therefore, $\sum \frac{\partial r}{\partial x} i = \sum (x/r)\, i$

$= 1/r \sum xi$

$= 1/r\,(xi + yj + zk)$

$= r / r$

b. $\nabla r^n = n\, r^{n-2}\, \mathbf{r}$

proof: $\nabla r^n = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) r^n$

$$= \left(\frac{\partial r^n}{\partial x} i + \frac{\partial r^n}{\partial y} j + \frac{\partial r^n}{\partial z} k\right)$$

$$= \left(i\, nr^{n-1} \frac{\partial r}{\partial x} + j\, nr^{n-1} \frac{\partial r}{\partial y} + k\, nr^{n-1} \frac{\partial r}{\partial z}\right)$$

$$= i\, nr^{n-1}\, x/r + j\, nr^{n-1}\, y/r + k\, nr^{n-1} z/r$$

[from 1, 2 and 3]

$$= nr^{n-2}(xi + yj + zk)$$

$$= nr^{n-2}\, \mathbf{r}$$

## DIVERGENCE OF A VECTOR FUNCTION:

If f(x, y, z) be a continuously differentiable vector function, then divergence of f is denoted by div f or ∇.f, and is defined by,

$$\text{Div f} = \nabla.\text{f} = \left(\frac{\partial}{\partial x}\text{i} + \frac{\partial}{\partial y}\text{ j} + \frac{\partial}{\partial z}\text{ k}\right).\text{f}$$

$$\nabla.\text{f} = \left(\frac{\partial f}{\partial x}\text{i} + \frac{\partial f}{\partial y}\text{ j} + \frac{\partial f}{\partial z}\text{ k}\right)$$

If $\quad f = f_1 i + f_2 j + f_3 k$, then

$$\nabla.\text{f} = \frac{\partial f_1}{\partial x}\text{i} + \frac{\partial f_2}{\partial y}\text{ j} + \frac{\partial f_3}{\partial z}\text{ k}$$

clearly, the divergence of a vector function f is a scalar function.

If f is a constant vector, then ∇.f= 0

A vector point function f is said to be solenoid, if ∇.f = 0 (div f = 0)

Clearly, a constant vector function is a solenoid.

## CURL OF A VECTOR FUNCTION:

If f is a vector function, continuously differentiable, then curl of f is denoted by curl f or ∇×f, and it is defined by,

$$\text{Curl } f = \nabla\times f = \frac{\partial}{\partial x}\times i + \frac{\partial}{\partial y}\times j + \frac{\partial}{\partial z}\times k$$

If $f = f_1 i + f_2 j + f_3 k$, then

$$\nabla\times f = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times (f_1 i + f_2 j + f_3 k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) i + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) k$$

Clearly, the curl of a vector point function f is again a vector point function.

if f is a constant function, then curl f = 0, i.e., ∇×f = 0

A vector function f is said to be irrational if ∇×f = 0

APPLICATION OF VECTOR CALCULUS:

Vector calculus has its applications in many fields , such as physics , engineering , biology , etc .., it also has a huge impact on our daily life - from microwave , cell phones , TV , and car to medicines , economy etc ..,

Some of the applications are given below:

1.Linear approximations are used to replace complicated functions with linear functions that are almost the same. Given a differentiable function f (x, y ) for( x , y) close to (a , b) by the formula,

$$f(x , y) \approx f(a , b ) + \partial f/\partial x(a , b)(x-a) + \partial f/\partial y(a , b) (y-b)$$

The right hand side is the equation of the plane tangent to the

graph of $z = f(x , y)$ at (a, b)

2.Credit card companies uses calculus to set the minimum payments due on credit card statements at the exact time.

- 3. Biologists use differential calculus to determine the exact rate of growth in a bacterial culture when different variables such as temperature and food source are changed.
- 4. A physics uses calculus to find the centre of mass, centre of mass of distribution of mass in space is the unique point where the weighted relative position of the distributed mass sums to zero. The distribution of mass is balanced around the centre of mass and the average of the weighted position coordinates of the distributed mass defines its coordinates.
- We can consider the child’s toy, which uses the principle of centre of mass to keep balance on a finger.

- 5.A graphics artist uses calculus to determine how different 3-dimensional models will behave when subjected to rapidly changing conditions. This can create a realistic environment for movies or videogames.